

ON STABLE COMPACTIFICATION WITH CASIMIR-LIKE POTENTIAL

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Multidimensional cosmological models with a higher dimensional space-time manifold $M = \mathbb{R} \times M_0 \times \prod_{i=1}^n M_i$ ($n > 1$) are investigated under dimensional reduction to D_0 -dimensional effective models. In the Einstein conformal frame, the effective potential for the internal scale factors is obtained. The stable compactification of the internal spaces is achieved due to the Casimir effect. In the case of more than one internal space a Casimir-like ansatz for the energy density of the massless scalar field fluctuations is proposed. Stable configurations with respect to the internal scale factor excitations are found in the cases of one and two internal spaces.

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1. Introduction

Our observable universe at the present time at large scales is well described by the Friedmann model with 4-dimensional Friedmann-Robertson-Walker metric. However, it is possible that space-time at short (Planck) distances might have a dimensionality of more than four and possess a rather complex topology [1]. Accounting this possibility it is natural to generalize the Friedmann model to a multidimensional cosmological model (MCM) with the topology [2], [3]

$$M = \mathbb{R} \times M_0 \times M_1 \times \dots \times M_n, \quad (1.1)$$

where M_i ($i = 0, \dots, n$) are spaces of constant curvature (more generally, Einstein spaces). M_0 usually denotes a d_0 -dimensional external space and M_i ($i = 1, \dots, n$) are d_i -dimensional internal spaces. These internal spaces have to be compact, what can be achieved by appropriate periodicity conditions for the coordinates [4]-[8]. As a result, internal spaces may have a nontrivial topology, being compact (i.e. closed and bounded) for any sign of the spatial curvature.

To make the internal dimensions unobservable at the present time the internal spaces have to be reduced to scales near the Planck length $L_{Pl} \sim 10^{-33}$ cm, i.e. scale factors of the internal spaces a_i should be of order of L_{Pl} . Obviously, such compactifications have to be stable. It means an effective potential of the model obtained under dimensional reduction to a 4-dimensional effective theory should have minima at $a_i \sim L_{Pl}$ ($i = 1, \dots, n$). A lot of papers were devoted to the problem of stable compactification of extra dimensions (an extended list of references may be found in the paper [9]). A number of effective potentials which ensure stability was obtained there. Among

them the Casimir potential is one of the most important. The Casimir effect is connected with the vacuum polarization of quantized fields due to non-trivial topology of the background space or the presence of boundaries in the space. As a result one obtains a nonvanishing energy density of the quantized fields in the vacuum state. In our case this phenomenon should take place due to internal spaces compactness.

The Casimir energy density for a massless scalar field in a model with one finite scale factor has a form [10]-[22]

$$\rho = C \frac{1}{a^D}, \quad (1.2)$$

where D is the total space-time dimension and C is a constant which depends strongly on the topology of the model. Equation (1.2) holds for models with one factor-space (e.g. M_0 with the scale factor $a_0 \equiv a$) [10], [11], [15], [16], [22], or in the case of one internal space and an external scale factor that is much greater than the internal one: $a_0 \gg a_1 \equiv a$ [12]-[14].

In the case of several internal compact spaces the Casimir effect occurs if at least one of them is finite. It is natural to consider all internal spaces on equal footing supposing that their scale factors are freezed near Planck length. In the papers [23, 24] an approximation of the Casimir energy density was proposed in the form

$$\rho = \frac{1}{\prod_{i=1}^n V_i} \sum_{i=1}^n \frac{A^{(i)}}{a_i^4} \quad (1.3)$$

for the universe with the topology $M = \mathbb{R} \times S^3 \times \prod_{i=1}^n S^d$ where $V_i \sim a_i^d$ is the volume of the d -dimensional sphere. It can be easily seen from this equation that $\rho \rightarrow 0$ if any of $a_i \rightarrow \infty$. This means that approximation (1.3) is not applicable in this limit because, first, for any finite a_i there should exist a nonvanishing Casimir energy density, and, second, it should provide a correct transition to Eq. (1.2), what, obviously, is not the case.

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The calculation of the Casimir energy density even in the case of one scale factor is a complicated problem. For several scale factors this procedure becomes extremely difficult, especially analytically. In the present paper we suggest ad hoc a Casimir-like ansatz for the energy density of the massless scalar field fluctuations, which yields a better approximation than (1.3). In this ansatz all internal scale factors are included on equal footing, it has the correct Casimir dimension: $[\text{cm}]^{-D}$ and gives a correct transition to the formula (1.2). The proposed ansatz gives an energy density that does not equal to zero if at least one of scale factors is finite.

As it was mentioned above the problem of stable compactification is one of the most important in MCM. In our paper we investigate this problem for the proposed Casimir-like potential in the case of one and two internal scale factors. It is shown that stable configurations exist for both of these cases. To our knowledge the investigation of the stable compactification problem due to Casimir potential for models with more than one non-identical (with respect to the scale factors) spaces was not performed up to now.

The paper is organized as follows. In Section 2, the general description of the considered model is given. In Section 3, the effective potential is obtained under dimensional reduction to a D_0 -dimensional (usually $D_0 = 4$) effective theory in the Einstein frame. The problem of stable compactification is investigated in Section 4 for one internal space and in the Section 5 for two internal spaces. Stable configurations are found for both of these cases. Some arguments in favour of our Casimir-like ansatz are listed in the Appendix A. In Appendix B, some general features of a generalization of the Abel-Plana summation formula to higher dimensional complex spaces are described. A simple example for a potential with nondegenerate minimum at a point with coinciding scale factors is given in Appendix C.

2. The model

Let us consider a cosmological model with the metric

$$ds^2 = -e^{2\gamma} d\tau^2 + \sum_{i=0}^n e^{2\beta^i(\tau)} g^{(i)}, \quad (2.1)$$

which is defined on the manifold (1.1), where the manifolds M_i with the metrics $g^{(i)}$ are Einstein spaces of dimension d_i , i.e.

$$R_{m_i n_i} [g^{(i)}] = \lambda^i g_{m_i n_i}^{(i)}, \quad i = 0, \dots, n \quad (2.2)$$

and

$$R[g^{(i)}] = \lambda^i d_i \equiv R_i. \quad (2.3)$$

In the case of constant curvature spaces $\lambda^i = k_i(d_i - 1)$, $k_i = \pm 1, 0$. The non-zero components of the Ricci

tensor for the metric (2.1) read

$$R_{00} = - \sum_{i=0}^n d_i \left[\ddot{\beta}^i - \dot{\gamma} \dot{\beta}^i + (\dot{\beta}^i)^2 \right], \quad (2.4)$$

$$R_{m_i n_i} = g_{m_i n_i}^{(i)} \left[\lambda^i + \exp(2\beta^i - 2\gamma) \times \right. \\ \left. \times \left(\ddot{\beta}^i + \dot{\beta}^i \left(\sum_{i=0}^n d_i \dot{\beta}^i - \gamma \right) \right) \right], \quad (2.5)$$

so that the corresponding scalar curvature is given as

$$R = \sum_{i=0}^n R_i \exp(-2\beta^i) + \exp(-2\gamma) \sum_{i=0}^n d_i \times \\ \times \left[2\ddot{\beta}^i - 2\dot{\gamma} \dot{\beta}^i + (\dot{\beta}^i)^2 + \dot{\beta}^i \sum_{j=0}^n d_j \dot{\beta}^j \right]. \quad (2.6)$$

The overdot in formulae (2.4) - (2.6) denotes differentiation with respect to time τ .

The action of the model we choose in the following form

$$S = \frac{1}{2\kappa^2} \int_M d^D x \sqrt{|g|} \{ R[g] - 2\Lambda \} + S_c + S_{YGH}, \quad (2.7)$$

where $D = 1 + \sum_{i=0}^n d_i$ is the total dimension of the space-time, Λ is a D -dimensional cosmological constant, κ^2 — D -dimensional gravitational constant and S_{YGH} is the standard York-Gibbons-Hawking boundary term. The Casimir effect is taken into account via the additional term S_c of the form [25]

$$S_c = - \int_M d^D x \sqrt{|g|} \rho(x), \quad (2.8)$$

where $\rho(x)$ is the Casimir energy density (A.11). For the model under consideration the Casimir energy is a function of the scale factors (see (A.28) - (A.30)). After dimensional reduction the action (2.7) reads

$$S = \frac{\mu}{\kappa^2} \int d\tau L \quad (2.9)$$

with Lagrangian L given as

$$L = \frac{1}{2} e^{-\gamma + \gamma_0} \sum_{i,j=0}^n G_{ij} \dot{\beta}^i \dot{\beta}^j + \frac{1}{2} e^{\gamma + \gamma_0} \sum_{i=0}^n R_i e^{-2\beta^i} \\ - e^{\gamma + \gamma_0} \Lambda - \frac{\kappa^2}{\mu} e^{\gamma} F_c. \quad (2.10)$$

Here $\gamma_0 = \sum_{i=0}^n d_i \beta^i$ and $\mu = \prod_{i=0}^n \mu_i$ where μ_i is defined by the equation (A.12). The components of the minisuperspace metric read [3]

$$G_{ij} = d_i \delta_{ij} - d_i d_j, \quad i, j = 0, \dots, n. \quad (2.11)$$

It is easy to show that the Euler-Lagrange equations for the Lagrangian (2.10) are equivalent to the Einstein equations for the metric (2.1) and the Casimir

energy-momentum tensor (A.14). This follows from the fact that for the non-trivial components of the tensor E_{MN} we get

$$E_{00} = \frac{\partial L}{\partial \gamma} \exp(\gamma - \gamma_0), \quad (2.12)$$

$$E_{m_i n_i} = \frac{1}{d_i} g_{m_i n_i}^{(i)} \left(\frac{d}{d\tau} \frac{\partial L}{\partial \beta^i} - \frac{\partial L}{\partial \beta^i} \right) \times \exp(2\beta^i - \gamma - \gamma_0), \quad (2.13)$$

$$i = 0, \dots, n,$$

where the tensor E_{MN} is defined as

$$E_{MN} = R_{MN} - \frac{1}{2} g_{MN} R - \kappa^2 T_{MN} + \Lambda g_{MN}. \quad (2.14)$$

For the proof one makes explicit use of equations (A.11) and (A.13).

3. The effective potential

Let us slightly generalize this model to the inhomogeneous case supposing that the scale factors $\beta^i = \beta^i(x)$ ($i = 0, \dots, n$) are functions of the coordinates x , where x are defined on the $D_0 = (1 + d_0)$ -dimensional manifold $\bar{M}_0 = \mathbb{R} \times M_0$ with the metric

$$\bar{g}^{(0)} = \bar{g}_{\mu\nu}^{(0)} dx^\mu \otimes dx^\nu = -e^{2\gamma} d\tau^2 + e^{2\beta^0(x)} g^{(0)}. \quad (3.1)$$

\bar{M}_0 denotes the external space-time. The dimensional reduction of the action (2.7) yields

$$S = \frac{1}{2\kappa_0^2} \int_{\bar{M}_0} d^{D_0} x \sqrt{|\bar{g}^{(0)}|} \prod_{i=1}^n e^{d_i \beta^i} \left\{ R[\bar{g}^{(0)}] - G_{ij} \bar{g}^{(0)\mu\nu} \partial_\mu \beta^i \partial_\nu \beta^j + \sum_{i=1}^n R_i e^{-2\beta^i} - 2\Lambda - 2\kappa^2 \rho \right\}, \quad (3.2)$$

where $\kappa_0^2 = \kappa^2/\mu$ is the D_0 -dimensional gravitational constant, $\mu = \prod_{i=1}^n \mu_i$ and G_{ij} ($i, j = 1, \dots, n$) is the midisuperspace metric with the components (2.11). Here the internal scale factors play the role of scalar fields (dilaton). The action (3.2) is written in the Brans-Dicke frame. Conformal transformation to the Einstein frame

$$\hat{g}_{\mu\nu}^{(0)} = \left(\prod_{i=1}^n e^{d_i \beta^i} \right)^{\frac{2}{D_0-2}} \bar{g}_{\mu\nu}^{(0)} \quad (3.3)$$

yields

$$S = \frac{1}{2\kappa_0^2} \int_{\bar{M}_0} d^{D_0} x \sqrt{|\hat{g}^{(0)}|} \left\{ \hat{R}[\hat{g}^{(0)}] - \bar{G}_{ij} \hat{g}^{(0)\mu\nu} \partial_\mu \beta^i \partial_\nu \beta^j - 2U_{eff} \right\}, \quad (3.4)$$

where the tensor components of the midisuperspace metric \bar{G}_{ij} are

$$\bar{G}_{ij} = d_i \delta_{ij} + \frac{1}{D_0 - 2} d_i d_j, \quad i, j = 1, \dots, n \quad (3.5)$$

and the effective potential U_{eff} reads

$$U_{eff} = \left(\prod_{i=1}^n e^{d_i \beta^i} \right)^{-\frac{2}{D_0-2}} \left[-\frac{1}{2} \sum_{i=1}^n R_i e^{-2\beta^i} + \Lambda + \kappa^2 \rho \right]. \quad (3.6)$$

In what follows we consider the case where the Casimir energy density depends on the internal scale factors: $\rho = \rho(\beta^1, \dots, \beta^n)$. This means that either the external space M_0 is non-compact or the external scale factor is much greater than the internal ones: $a_0 \gg a_1, \dots, a_n$.

In the case of one internal space $n = 1$ the action and the effective potential are respectively

$$S = \frac{1}{2\kappa_0^2} \int_{\bar{M}_0} d^{D_0} x \sqrt{|\hat{g}^{(0)}|} \left\{ \hat{R}[\hat{g}^{(0)}] - \hat{g}^{(0)\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 2U_{eff} \right\} \quad (3.7)$$

and

$$U_{eff} = e^{2\varphi \left[\frac{d_1}{(D-2)(D_0-2)} \right]^{1/2}} \left[-\frac{1}{2} R_1 e^{2\varphi \left[\frac{D_0-2}{d_1(D-2)} \right]^{1/2}} + \Lambda \kappa^2 \rho(\varphi) \right], \quad (3.8)$$

where we redefined the dilaton field as

$$\varphi \equiv -\sqrt{\frac{d_1(D-2)}{D_0-2}} \beta^1. \quad (3.9)$$

The minima of the effective potentials (3.6) and (3.8) define stable compactification positions $a_{(c)i} = \exp \beta_c^i$, $c = 1, \dots, m$ and small excitations ψ^i of the internal scale factors near these positions have a form of minimally coupled massive scalar fields in the external space-time [9]:

$$S = \frac{1}{2\kappa_0^2} \int_{\bar{M}_0} d^{D_0} x \sqrt{|\hat{g}^{(0)}|} \left\{ \hat{R}[\hat{g}^{(0)}] - 2\Lambda_{(c)eff} \right\} + \sum_{i=1}^n \frac{1}{2} \int_{\bar{M}_0} d^{D_0} x \sqrt{|\hat{g}^{(0)}|} \left\{ -\hat{g}^{(0)\mu\nu} \psi_{,\mu}^i \psi_{,\nu}^i - m_{(c)i}^2 \psi^i \psi^i \right\}, \quad (3.10)$$

where $\Lambda_{(c)eff} \equiv U_{eff}(\vec{\beta}_c)$ is an effective cosmological constant and $m_{(c)i}$ are masses of gravitational excitons corresponding to the c -th minimum. From a physical point of view it is clear that the effective potential should satisfy following conditions:

- (i) $a_{(c)i} \gtrsim L_{Pl}$,
- (ii) $m_{(c)i} \leq M_{Pl}$,
- (iii) $\Lambda_{(c)eff} \rightarrow 0$.

The first condition expresses the fact that the internal spaces should be unobservable at the present time and stable against quantum gravitational fluctuations. This condition ensures the applicability of the classical gravitational equations near positions of minima of the effective potential. The second condition means that the curvature of the effective potential should be less than Planckian one. Of course, gravitational excitons can be excited at the present time if $m_i \ll M_{Pl}$. The third condition reflects the fact that the cosmological constant at the present time is very small: $\Lambda \leq 10^{-56} \text{cm}^{-2} \approx 10^{-121} \Lambda_{Pl}$, where $\Lambda_{Pl} = L_{Pl}^{-2}$. Thus, for simplicity, we can demand $\Lambda_{eff} = U_{eff}(\vec{\beta}_c) = 0$. (We used the abbreviation $\Lambda_{eff} \equiv \Lambda_{(c)eff}$.) Strictly speaking, in the multi-minimum case ($c > 1$) we can demand $a_{(c)i} \sim L_{Pl}$ and $\Lambda_{(c)eff} = 0$ only for one of the minima to which corresponds the present universe state. For all other minima it may be $a_{(c)i} \gg L_{Pl}$ and $|\Lambda_{(c)eff}| \gg 0$.

4. One internal space

Here we consider the case of one internal space $n = 1$ or, strictly speaking, the case where all internal spaces have one common scale factor: $a_1 \equiv a_2 \equiv \dots \equiv a_n \equiv a = \exp \beta$. The latter case corresponds to the splitting of the internal space into a product of Einstein spaces [26]: $M_1 \rightarrow \prod_{i=1}^n M_i$ which leads in the equations (3.7) - (3.9) to the substitutions: $d_1 \rightarrow d = \sum_{i=1}^n d_i$; $R_1 \rightarrow R = \sum_{i=1}^n R_i$. For effective one-scale-factor models the Casimir energy density reads according to (A.26): $\rho = C \exp(-D\beta)$, where C is a constant that strongly depends on the topology of the model. For example, for fluctuations of massless scalar fields the constant C was calculated to take the values: $C = -8.047 \cdot 10^{-6}$ if $\bar{M}_0 = \mathbb{R} \times S^3$, $M_1 = S^1$ (with e^{β^0} as scale factor of S^3 and $e^{\beta^0} \gg e^{\beta^1}$) [14]; $C = -1.097$ if $\bar{M}_0 = \mathbb{R} \times \mathbb{R}^2$, $M_1 = S^1$ [12] and $C = 3.834 \cdot 10^{-6}$ if $\bar{M}_0 = \mathbb{R} \times S^3$, $M_1 = S^3$ (with $e^{\beta^0} \gg e^{\beta^1}$) [14].

For an effective potential with $\rho = C \exp(-D\beta)$ the zero-extremum-conditions $\frac{\partial U_{eff}}{\partial \beta} \Big|_{min} = 0$ and $\Lambda_{eff} = 0$ lead to a fine tuning of the parameters of the model

$$R_1 e^{-2\beta_c} = \frac{2D}{D-2} \Lambda, \quad R_1 e^{(D-2)\beta_c} = \kappa^2 C D, \quad (4.1)$$

which implies $\text{sign } R_1 = \text{sign } \Lambda = \text{sign } C$. We note that a similar fine tuning was obtained by different methods in papers [25] (for one internal space) and [27] (for n identical internal spaces). To get a minimum, the second derivative of the effective potential at the extremum should be positive:

$$\frac{\partial^2 U_{eff}}{\partial \beta^2} \Big|_{\beta_c} = (D-2) R_1 (e^{-2\beta_c})^{\frac{D-2}{D_0-2}} > 0. \quad (4.2)$$

Thus, a stable compactification takes place if the internal space has a positive scalar curvature $R_1 > 0$ (or for a split space M_1 the sum of the curvatures of the constituent spaces M_1^k should be positive). The mass squared of the gravitational excitons is given by the expression

$$m^2 \equiv \frac{\partial^2 U_{eff}}{\partial \varphi^2} \Big|_{\varphi_c} = \frac{D_0-2}{d_1} R_1 (e^{-2\beta_c})^{\frac{D-2}{D_0-2}}. \quad (4.3)$$

As example, let us consider a manifold M with topology $M = \mathbb{R} \times S^3 \times S^3$ where $a_0 \gg a_1$. Then according to [14] the constant C is given as $C = 3.834 \cdot 10^{-6} > 0$, so that with $C, R_1 = d_1(d_1-1) = 6 > 0$ the effective potential has a minimum provided $\Lambda > 0$. Normalizing κ_0^2 to unity, we get $\kappa^2 = \mu$, where $\mu = 2\pi^{(d+1)/2} / \Gamma(\frac{1}{2}(d+1))$ is the volume of the d -dimensional sphere. For the model with the topology $M = \mathbb{R} \times S^3 \times S^3$ we obtain $a_c \approx 1.5 \cdot 10^{-1} L_{Pl}$ and $m \approx 2.12 \cdot 10^2 M_{Pl}$. Hence, for this particular example the conditions (i) and (ii) are not satisfied. It would be interesting to investigate models with more complex topology. Of course, the calculation of the Casimir effect in this case becomes a very complicated problem. Therefore in the Appendix A we propose ad hoc the Casimir-like expressions (A.28), (A.29) for the energy density in the case of more than one internal spaces with non-identical scale factors. In the next section we investigate the stable compactification problem for this potential in the case of two internal spaces. For this case the Casimir-like potential considerably simplifies, but even in this case the stability analysis is still complicated. To our knowledge the investigation of the stable compactification problem for models with more than one non-identical (with respect to the scale factors) spaces was not performed up to now.

5. Two internal spaces

After these brief considerations on Casimir potentials for one-scale-factor models we turn now to some methods applicable for an analysis of two-scale-factor models with Casimir-like potentials. Some arguments in favour of such potentials are given in Appendix A. We propose to use these potentials of the general form

$$\begin{aligned} \rho = & e^{-\sum_{i=0}^n d_i \beta^i} \sum_{k_0, \dots, k_n=0}^n \epsilon_{k_0|k_1 \dots k_n} \sum_{\xi_0=0}^{d_{k_1}} \sum_{\xi_1=0}^{d_{k_2}} \dots \\ & \dots \sum_{\xi_{n-1}=0}^{d_{k_n}} A_{\xi_0 \dots \xi_{n-1}}^{(k_0)} \frac{(e^{\beta^{k_1}})^{\xi_0} \dots (e^{\beta^{k_n}})^{\xi_{n-1}}}{(e^{\beta^{k_0}})^{\xi_0 + \xi_1 + \dots + \xi_{n-1} + 1}}, \end{aligned} \quad (5.1)$$

in order to achieve a first crude insight into a possible stabilization mechanism of internal space configurations due to exact Casimir potentials depending on

n scale factors. In (5.1) $A_{\xi_0 \dots \xi_{n-1}}^{(k_0)}$ are dimensionless constants which depend on the topology of the model and $\epsilon_{ik \dots m} = (-1)^P \varepsilon_{ik \dots m}$. Here $\varepsilon_{ik \dots m}$ is the totally antisymmetric symbol ($\varepsilon_{01 \dots m} = +1$), P is the number of the permutations of the $01 \dots n$ resulting in $ik \dots m$. $|k_1 k_2 \dots k_n|$ means summation is taken over $k_1 < k_2 < \dots < k_n$.

From investigations performed in the last decades (see e.g. [16, 17], Refs. therein and Appendix A below) we know that exact Casimir potentials can be expressed in terms of Epstein zeta function series with scale factors as parameters. Unfortunately, the existing representations of these zeta function series are not well suited for a stability analysis of the effective potential U_{eff} as function over the total target space $\beta \in \mathbb{R}_T^n$. The problems can be circumvented partially by the use of asymptotic expansions of the zeta function series in terms of elementary functions for special subdomains Ω_a of the target space $\Omega_a \subset \mathbb{R}_T^n$. According to [16, 17] potential (5.1) gives a crude approximation of exact Casimir potentials in subdomains Ω_a . In contrast with other approximative potentials proposed in literature [23, 24] potential (5.1) shows a physically correct behavior under decompactification of factor-space components. The question, in as far (5.1) can be used in regions $\mathbb{R}_T^n \setminus \Omega_a$, needs an additional investigation. The philosophy of the proposed method consists in a consideration of potentials (5.1) on the whole target space \mathbb{R}_T^n , and testing of scale factors and parameters of possible minima of the corresponding effective potential on their compatibility with asymptotic approximations of exact Casimir potentials in Ω_a . As a beginning, we describe in the following only some techniques, without explicit calculation and estimation of exciton masses.

Before we start our analysis of two-scale-factor models with Casimir-like potentials

$$\rho = e^{-\sum_{i=1}^2 d_i \beta^i} \left[\sum_{i=0}^{d_2} A_i^{(1)} \frac{e^{i\beta^2}}{e^{(D_0+i)\beta^1}} + \sum_{j=0}^{d_1} A_j^{(2)} \frac{e^{j\beta^1}}{e^{(D_0+j)\beta^2}} \right] \quad (5.2)$$

let us introduce the following convenient (temporary) notations: $x := a_1 \equiv e^{\beta^1}$, $y := a_2 \equiv e^{\beta^2}$, $P_\xi := \kappa^2 A_\xi^{(1)}$, $S_\xi := \kappa^2 A_\xi^{(2)}$. In terms of these notations the effective potential (3.6) reads

$$U_{eff} = (x^{d_1} y^{d_2})^{-\frac{2}{D_0-2}} \left[-\frac{R_1}{2} x^{-2} - \frac{R_2}{2} y^{-2} + \Lambda + x^{-d_1} y^{-d_2} \left(\sum_{i=0}^{d_2} P_i y^i x^{-(D_0+i)} + \sum_{j=0}^{d_1} S_j x^j y^{-(D_0+j)} \right) \right]. \quad (5.3)$$

For physically relevant configurations with scale factors near Planck length

$$0 < x, y < \infty \quad (5.4)$$

we transform extremum conditions $\partial_{\beta^{1,2}} U_{eff} = 0 \Leftrightarrow \partial_{x,y} U_{eff} = 0$ by factoring out of $(xy)^{-D}$ -terms and taking combinations $\partial_x U_{eff} \pm \partial_y U_{eff} = 0$ to an equivalent system of two algebraic equations in x and y :

$$\begin{aligned} I_{1+} &= (xy)^{D-2} \left[\frac{D-2}{D_0-2} (R_1 y^2 + R_2 x^2) - \frac{2\Lambda}{D_0-2} D' x^2 y^2 \right] - \left(\frac{2D'}{D_0-2} + D \right) \times \\ &\times \left[\sum_{i=0}^{d_2} P_i y^{D_0+d_1+i} x^{d_2-i} + \sum_{j=0}^{d_1} S_j y^{d_1-j} x^{D_0+d_2+j} \right] \\ &= 0 \\ I_{1-} &= (xy)^{D-2} \left[\frac{d_1-d_2}{D_0-2} (R_1 y^2 + R_2 x^2) + (R_1 y^2 - R_2 x^2) - \frac{2\Lambda}{D_0-2} (d_1 - d_2) x^2 y^2 \right] - \\ &- \sum_{i=0}^{d_2} P_i \left[D_0 \left(\frac{d_1-d_2}{D_0-2} + 1 \right) + 2i \right] y^{D_0+d_1+i} x^{d_2-i} - \\ &- \sum_{j=0}^{d_1} S_j \left[D_0 \left(\frac{d_1-d_2}{D_0-2} - 1 \right) - 2j \right] y^{d_1-j} x^{D_0+d_2+j} \\ &= 0. \end{aligned} \quad (5.5)$$

(Here we used the notation $D' := d_1 + d_2$.) Thus, scale factors a_1 and a_2 satisfying the extremum conditions are defined as common roots of polynomials (5.5). In the general case of arbitrary dimensions (D_0, d_1, d_2) and arbitrary parameters $\{R_1, R_2, P_i, S_i\}$ these roots are complex, so that only a restricted subclass of them are real and fulfill condition (5.4). In the following we derive necessary conditions on the parameter set guaranteeing the existence of real roots satisfying (5.4). The analysis could be carried out using resultant techniques [18] on variables x, y directly. The structure of $I_{1\pm}$ suggests another, more convenient method [19]. Introducing the projective coordinate $\lambda = y/x$ we rewrite (5.5) as $I_{1\pm} = x^D I_{2\pm}(\lambda)$ with

$$I_{2+} = -a_0(\lambda) + a_{D-2}(\lambda) y^{D-2} - y^D \Delta_+ = 0 \quad (a) \quad (5.6)$$

$$I_{2-} = -b_0(\lambda) + b_{D-2}(\lambda) y^{D-2} - y^D \Delta_- = 0 \quad (b)$$

and coefficient-functions

$$\begin{aligned} a_0(\lambda) &= \left[\frac{2D'}{D_0-2} + D \right] \left[\sum_{i=0}^{d_2} P_i \lambda^{D_0+d_1+i} + \sum_{j=0}^{d_1} S_j \lambda^{d_1-j} \right] \\ a_{D-2}(\lambda) &= \frac{D-2}{D_0-2} (R_1 \lambda^2 + R_2) \end{aligned}$$

$$\begin{aligned}
b_0(\lambda) &= \sum_{i=0}^{d_2} P_i \left[D_0 \left(\frac{d_1 - d_2}{D_0 - 2} + 1 \right) + 2i \right] \lambda^{D_0 + d_1 + i} + \\
&\quad + \sum_{j=0}^{d_1} S_j \left[D_0 \left(\frac{d_1 - d_2}{D_0 - 2} - 1 \right) - 2j \right] \lambda^{d_1 - j} \\
b_{D-2}(\lambda) &= \frac{d_1 - d_2}{D_0 - 2} (R_1 \lambda^2 + R_2) + (R_1 \lambda^2 - R_2) \\
\Delta_{\pm} &= \frac{2\Lambda}{D_0 - 2} (d_1 \pm d_2). \tag{5.7}
\end{aligned}$$

Equations (5.6) have common roots if the coefficient functions $\{a_i(\lambda), b_i(\lambda)\}$ are connected by a constraint. This constraint is given by the vanishing resultant

$$R_y[I_{2+}, I_{2-}] = w(\lambda) = 0. \tag{5.8}$$

Now, the roots can be obtained in two steps. First, one finds the set of roots $\{\lambda_i\}$ of the polynomial $w(\lambda)$. Physical condition (5.4) on the affine coordinates (x, y) implies here a corresponding condition on the projective coordinate $\lambda = y/x$

$$Im(\lambda) = 0, \quad 0 < \lambda < \infty. \tag{5.9}$$

Second, one searches for each λ_i solutions $\{y_{ij}\}$ of (5.6). The complete set of physically relevant solutions of system (5.5) is then given in terms of pairs $\{x_{ij} = y_{ij}/\lambda_i, y_{ij}\}$.

Because of the simple y -structure of equations (5.6) the polynomial $w(\lambda)$ can be derived from (5.6) directly, without explicit calculation of the resultant. Taking $b_0(\lambda)I_{2+} - a_0(\lambda)I_{2-} = 0$, $\Delta_- I_{2+} - \Delta_+ I_{2-} = 0$ and assuming $y > 0$ we get

$$y^2 = \frac{L_3}{L_1}, \quad y^{D-2} = \frac{L_1}{L_2}, \tag{5.10}$$

where

$$\begin{aligned}
L_1(\lambda) &:= \Delta_- a_0(\lambda) - \Delta_+ b_0(\lambda) \\
L_2(\lambda) &:= \Delta_- a_{D-2}(\lambda) - \Delta_+ b_{D-2}(\lambda) \\
L_3(\lambda) &:= a_0(\lambda)b_{D-2}(\lambda) - b_0(\lambda)a_{D-2}(\lambda) \tag{5.11}
\end{aligned}$$

depend only on λ . Excluding y from (5.10) yields the necessary constraint for the coefficient functions of equation system (5.6)

$$w(\lambda) = L_2^2(\lambda)L_3^{D-2}(\lambda) - L_1^D(\lambda) = 0. \tag{5.12}$$

Together with condition (5.9), this polynomial of degree

$$\deg_{\lambda}[w(\lambda)] = D^2 \tag{5.13}$$

can be used for a first test of internal space configurations on stability of their compactification. If the corresponding parameters $\{R_1, R_2, P_i, S_i\}$ allow the existence of positive real roots λ_i , the space configuration is a possible candidate for a stable compactified configuration and can be further tested on the existence

of minima of the effective potential U_{eff} . Otherwise it belongs to the class of unstable internal space configurations.

Before we turn to the consideration of two-scale-factor models with factor-spaces of the same topological type ($M_1 = M_2$) we note that for the coefficient functions (5.11), because of (5.4) and (5.10), there must hold

$$\text{sign}(L_1)|_{\lambda_i} = \text{sign}(L_2)|_{\lambda_i} = \text{sign}(L_3)|_{\lambda_i}. \tag{5.14}$$

Furthermore we see from (5.12) that for even dimensions $D = \dim(M_1) + \dim(M_2) + \dim(M_0) + 1$ of the product-manifold the polynomial $w(\lambda)$ factors into two subpolynomials of degree $D^2/2$

$$\begin{aligned}
w(\lambda) &= \left[L_2(\lambda)L_3^{\frac{D-2}{2}}(\lambda) + L_1^{\frac{D}{2}}(\lambda) \right] \times \\
&\quad \times \left[L_2(\lambda)L_3^{\frac{D-2}{2}}(\lambda) - L_1^{\frac{D}{2}}(\lambda) \right] \\
&= 0. \tag{5.15}
\end{aligned}$$

5.1. Two identical internal factor-spaces

In the case of identical internal factor-spaces M_1 and M_2 we have $d_1 = d_2, P_i = S_i, R_1 = R_2$. If we assume additionally an external space-time \bar{M}_0 with $\dim \bar{M}_0 = 4$ and, hence, $D = 2(d_1 + 2)$, then equations (5.6) and polynomial (5.15) can be rewritten as

$$\begin{aligned}
I_{2+} &= -4(d_1 + 1)\bar{a}_0(\lambda) + (d_1 + 1)R_1(\lambda^2 + 1)y^{D-2} - \\
&\quad - 2d_1\Lambda y^D = 0 \tag{a} \\
I_{2-} &= (\lambda^2 - 1) \underbrace{[-2\bar{b}_0(\lambda) + R_1 y^{D-2}]}_{\bar{I}_{2-}} = 0 \tag{b}
\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}
w(\lambda) &= 4\Lambda^2 d_1^2 [2(\lambda^2 - 1)]^{2(d_1+2)} \times \\
&\quad \times \underbrace{\left[R_1^{d_1+2} (2(d_1 + 1))^{d_1+1} \bar{L}_3^{d_1+1} + (-2\Lambda d_1)^{d_1+1} \bar{b}_0^{d_1+2} \right]}_{w_+(\lambda)} \\
&\quad \times \underbrace{\left[R_1^{d_1+2} (2(d_1 + 1))^{d_1+1} \bar{L}_3^{d_1+1} - (-2\Lambda d_1)^{d_1+1} \bar{b}_0^{d_1+2} \right]}_{w_-(\lambda)} \\
&= 0 \tag{5.17}
\end{aligned}$$

with the notations

$$\begin{aligned}
\bar{L}_3 &:= 2\bar{a}_0 - (\lambda^2 + 1)\bar{b}_0 \\
\bar{a}_0 &:= \sum_{i=0}^{d_1} P_i [\lambda^{4+d_1+i} + \lambda^{d_1-i}] \\
\bar{b}_0 &:= \sum_{i=0}^{d_1} P_i (2+i) \lambda^{d_1-i} \sum_{j=0}^{i+1} \lambda^{2j}. \tag{5.18}
\end{aligned}$$

According to the factorizing polynomial (5.16(b)) and the constraint (5.17) the root set splits into two types of subsets defined by the conditions

type I:

$$I_{2+}(\lambda, y) = 0, \quad \bar{I}_{2-}(\lambda, y) = 0, \quad w_+(\lambda)w_-(\lambda) = 0$$

type II:

$$I_{2+}(\lambda, y) = 0, \quad \lambda^2 - 1 = 0. \quad (5.19)$$

For roots of type I we have similar to (5.10)

$$y^2 = -\frac{(d_1+1)R_1\bar{L}_3}{2\Lambda d_1\bar{b}_0}, \quad y^{D-2} = \frac{2\bar{b}_0}{R_1}, \quad (5.20)$$

whereas type II roots at $\lambda = 1$ should be found from the polynomial I_{2+} (5.16(a)) directly. For this polynomial we have now simply

$$\begin{aligned} I_{2+}(\lambda = 1) &:= \frac{\Lambda d_1}{d_1 + 1} y^{2(d_1+2)} - R_1 y^{2(d_1+1)} + 4 \sum_{i=0}^{d_1} P_i \\ &= 0. \end{aligned} \quad (5.21)$$

Coming back to the general case of identical factor-spaces M_1, M_2 with coinciding or noncoinciding scale factors we note that there exists an interchange symmetry between M_1 and M_2 , which becomes apparent in the root structure of the polynomial $w(\lambda)$. From

$$\begin{aligned} U_{eff} &= (xy)^{-d_1} \left[-\frac{R_1}{2} (x^{-2} - y^{-2}) + \Lambda + \right. \\ &\quad \left. + (xy)^{-d_1} \sum_{i=0}^{d_1} P_i (y^i x^{-4-i} + x^i y^{-4-i}) \right] \end{aligned} \quad (5.22)$$

we see that x and y enter (5.22) symmetrically. When one extremum of (5.22) is located at $\{x_i = a, y_i = b\}$ then because of the interchange symmetry $x \rightleftharpoons y$ there exists a second extremum located at $\{x_j = b, y_j = a\}$. So we have for the corresponding projective coordinates :

$$\begin{aligned} \lambda_i &= y_i/x_i = b/a, \quad \lambda_j = y_j/x_j = a/b \\ \implies \lambda_i &= \lambda_j^{-1}. \end{aligned} \quad (5.23)$$

By regrouping of terms in (5.17) it is easy to show that

$$w(\lambda^{-1}) = \lambda^{-D^2} w(\lambda) \quad (5.24)$$

and, hence, roots $\{\lambda_i \neq 0\}$ of $w(\lambda) = 0$ exist indeed in pairs $\{\lambda_i, \lambda_i^{-1}\}$. But there is no relation connecting this root-structure with a symmetry between $w_+(\lambda)$ and $w_-(\lambda)$ in (5.17) $w_+(\lambda^{-1}) \not\sim w_-(\lambda)$. For completeness, we note that relation (5.23) is formally similar to dualities recently investigated in superstring theory [20].

Before we turn to an analysis of minimum conditions for effective potentials U_{eff} corresponding to

special classes of solutions of $w(\lambda) = 0$ we rewrite the necessary second derivatives

$$\begin{aligned} \partial_{xx}^2 U_{eff} &= -\frac{R_1}{2} (\alpha_1 x^{-d_1-4} y^{-d_1} + \\ &\quad + \alpha_2 x^{-d_1-2} y^{-d_1-2}) + \\ &\quad + \Lambda \alpha_2 x^{-d_1-2} y^{-d_1} + \\ &\quad + \sum_{i=0}^{d_1} P_i (\alpha_3 y^{i-2d_1} x^{-i-2d_1-6} + \\ &\quad + \alpha_4 x^{i-2d_1-2} y^{-i-2d_1-4}) \\ \partial_{yy}^2 U_{eff} &= \partial_{xx}^2 U_{eff} \Big|_{x \rightleftharpoons y} \\ \partial_{xy}^2 U_{eff} &= -\frac{R_1}{2} \alpha_5 (x^{-d_1-3} y^{-d_1-1} + \\ &\quad + x^{-d_1-1} y^{-d_1-3}) + \\ &\quad + \Lambda \alpha_6 x^{-d_1-1} y^{-d_1-1} + \\ &\quad + \sum_{i=0}^{d_1} P_i \alpha_7 (y^{i-2d_1-1} x^{-i-2d_1-5} + \\ &\quad + \alpha_4 x^{i-2d_1-1} y^{-i-2d_1-5}), \end{aligned} \quad (5.25)$$

where

$$\begin{aligned} \alpha_1 &= (d_1+2)(d_1+3) \\ \alpha_2 &= d_1(d_1+1) \\ \alpha_3 &= (2d_1+i+4)(2d_1+i+5) \\ \alpha_4 &= (2d_1-i)(2d_1-i+1) \\ \alpha_5 &= d_1(d_1+2) \\ \alpha_6 &= d_1^2 \\ \alpha_7 &= (2d_1+i+4)(2d_1-i), \end{aligned} \quad (5.26)$$

in the more appropriate form (notation $\tilde{\mu} = \lambda^{d_1} y^{-2D+2}$)

$$\begin{aligned} \partial_{xx}^2 U_{eff} &= \lambda^2 \tilde{\mu} \left[-\frac{R_1}{2} (\alpha_1 \lambda^2 + \alpha_2) y^{D-2} + \Lambda \alpha_2 y^D \right. \\ &\quad \left. + \sum_{i=0}^{d_1} P_i (\alpha_3 \lambda^{4+d_1+i} + \alpha_4 \lambda^{d_1-i}) \right] \\ \partial_{yy}^2 U_{eff} &= \tilde{\mu} \left[-\frac{R_1}{2} (\alpha_1 + \alpha_2 \lambda^2) y^{D-2} + \Lambda \alpha_2 y^D \right. \\ &\quad \left. + \sum_{i=0}^{d_1} P_i (\alpha_4 \lambda^{4+d_1+i} + \alpha_3 \lambda^{d_1-i}) \right] \\ \partial_{xy}^2 U_{eff} &= \lambda \tilde{\mu} \left[-\frac{R_1}{2} \alpha_5 (\lambda^2 + 1) y^{D-2} + \Lambda \alpha_6 y^D \right. \\ &\quad \left. + \sum_{i=0}^{d_1} P_i \alpha_7 (\lambda^{4+d_1+i} + \lambda^{d_1-i}) \right]. \end{aligned} \quad (5.27)$$

Introducing the notations

$$\tilde{A}_c := \begin{pmatrix} \partial_{xx}^2 U_{eff} & \partial_{xy}^2 U_{eff} \\ \partial_{xy}^2 U_{eff} & \partial_{yy}^2 U_{eff} \end{pmatrix} \quad (5.28)$$

and

$$w_{(c)1,2} := \frac{1}{2} \left[\text{Tr}(\tilde{A}_c) \pm \sqrt{\text{Tr}^2(\tilde{A}_c) - 4 \det(\tilde{A}_c)} \right] \quad (5.29)$$

the minimum conditions are given as

$$w_{(c)1} > 0, \quad w_{(c)2} \geq 0. \quad (5.30)$$

In the degenerate case of coinciding scale factors $x = y$, $\lambda = 1$ there hold the following relations between the derivatives of effective potentials $U_{eff}(x, y)$ and $\tilde{U}_{eff}(y) = U_{eff}(y, y)$

$$\begin{aligned} \partial_y \tilde{U}_{eff} &= \partial_x U_{eff}|_{x=y} + \partial_y U_{eff}|_{x=y} \\ \partial_{yy}^2 \tilde{U}_{eff} &= \partial_{xx}^2 U_{eff}|_{x=y} + \partial_{yy}^2 U_{eff}|_{x=y} \\ &\quad + 2\partial_{xy}^2 U_{eff}|_{x=y} \end{aligned} \quad (5.31)$$

and minimum conditions reduce to

$$\partial_y \tilde{U}_{eff} = 0, \quad \partial_{yy}^2 \tilde{U}_{eff} > 0 \quad (5.32)$$

with

$$\begin{aligned} \partial_{yy}^2 \tilde{U}_{eff} &= 2y^{-2D+2} [-R_1(d_1+1)(2d_1+3)y^{D-2} \\ &\quad + \Lambda d_1(2d_1+1)y^D \\ &\quad + 4(d_1+1)(4d_1+5) \sum_{i=0}^{d_1} P_i] \end{aligned} \quad (5.33)$$

For convenience of the additional explicit calculations of constraint $U_{eff}|_{\min} = 0$ we rewrite also effective potential (5.22) in terms of variables y , λ

$$\begin{aligned} U_{eff} &= \lambda^{d_1} y^{-2D+4} \left[-\frac{R_1}{2} y^{D-2} (\lambda^2 + 1) + \Lambda y^D + \right. \\ &\quad \left. + \sum_{i=0}^{d_1} P_i (\lambda^{4+d_1+i} + \lambda^{d_1-i}) \right] \end{aligned} \quad (5.34)$$

The further analysis consists in a compatibility consideration of minimum conditions (5.30) and (5.32) with properties of the polynomial $w(\lambda)$, expressions like (5.20) defining y^{D-2} and $y^D = y^{D-2} y^2$ as functions of λ on the parameter-space $\mathbb{R}_{par}^{d_1+3} = \{(R_1, \Lambda, P_i) \mid i = 0, \dots, d_1\}$ and the constraint $U_{eff}|_{\min} = 0$. As result we will get a first crude division of $\mathbb{R}_{par}^{d_1+3}$ in stability-domains allowing the existence of minima of the effective potential U_{eff} and forbidden regions corresponding to unstable internal space configurations.

After these general considerations we turn now to a more concrete analysis.

5.2. Noncoinciding scale factors

($\lambda \neq 1$), $R_1, \Lambda \neq 0$

In this case we have to consider roots of type I (5.19). We start from the polynomial $w(\lambda)$ knowing that stable internal space-configurations correspond to real projective coordinates $0 < \lambda < \infty$. So we have to test subpolynomials $w_{\pm}(\lambda)$ on the existence of such roots. The high degree $\deg_{\lambda}[w_{\pm}(\lambda)] = 2(d_1+2)(d_1+1) \geq 24$, $d_1 \geq 2$ (because of nonvanishing curvature of the factor-spaces M_1, M_2) allows only an analysis by techniques of number theory [21], the theory of ideals of commutative rings [18] or, for general parameter-configurations, numerical tests. In the latter case the number of effective test-parameters can be reduced by introduction of new coordinates in parameter-space

$$\begin{aligned} \mathbb{R}_{par}^{d_1+3} &\rightarrow \overline{\mathbb{R}}_{par}^{d_1+1} \\ \overline{\mathbb{R}}_{par}^{d_1+1} &= \left\{ (\chi, p_i) \mid \chi = \left(\frac{2\Lambda d_1}{d_1+1} \right)^{d_1+1} \frac{2P_0}{R_1^{d_1+2}}, \right. \\ &\quad \left. p_i = \frac{P_i}{P_0}, i = 1, \dots, d_1 \right\} \end{aligned} \quad (5.35)$$

(for $P_0 \neq 0$, $p_0 = 1$; in the opposite case P_0 can be replaced by any nonzero P_i). Polynomials $w_{\pm}(\lambda) = 0$ transform then to

$$w_{\pm}(\lambda) = \frac{1}{2} R_1^{d_1+2} ((d_1+1)P_0)^{d_1+1} \bar{w}_{\pm}(\lambda) = 0, \text{ where}$$

$$\bar{w}_{\pm}(\lambda) := \tilde{L}_3^{d_1+1} \pm (-)^{d_1+1} \chi \tilde{b}_0^{d_1+2} = 0$$

$$\tilde{L}_3 := \bar{L}_3(P_0 = 1; P_1 = p_1, \dots, P_{d_1} = p_{d_1}) \quad (5.36)$$

$$\tilde{b}_0 := \bar{b}_0(P_0 = 1; P_1 = p_1, \dots, P_{d_1} = p_{d_1}).$$

Test are easy to perform with programs like MATHEMATICA or MAPLE.

As a second step we have to consider minimum conditions (5.30). Using (5.20) we substitute

$$y^{D-2} = \frac{2\tilde{b}_0}{R_1}; \quad y^D = -\frac{(d_1+1)\tilde{L}_3}{\Lambda d_1} \quad (5.37)$$

into (5.27) and transform (5.30) to the following equivalent inequalities

$$\begin{aligned} &2(d_1+1)(d_1+2)Q_1(\lambda) + Q_2(\lambda) > \\ &> (d_1+2)(\lambda^2+1)\bar{b}_0(\lambda) \\ &[2(d_1+2)Q_1(\lambda) - (\lambda^2+1)\bar{b}_0(\lambda)] \times \\ &\times [Q_2(\lambda) - (\lambda^2+1)\bar{b}_0(\lambda)] \geq \\ &\geq (d_1+1)(\lambda^2-1)^2 \bar{b}_0^2(\lambda) \end{aligned} \quad (5.38)$$

with notations

$$Q_1(\lambda) := \sum_{i=0}^{d_1} P_i (\lambda^{4+d_1+i} + \lambda^{d_1-i}) \equiv \bar{a}_0(\lambda) \quad (5.39)$$

$$Q_2(\lambda) := \sum_{i=0}^{d_1} P_i (i+2)^2 (\lambda^{4+d_1+i} + \lambda^{d_1-i}).$$

Stability-domains in parameter-space $\mathbb{R}_{par}^{d_1+3}$, corresponding to minima of the effective potential are given as intersections of domains defined by (5.38) with domains which allow the existence of physical relevant roots of $\bar{w}_\pm(\lambda) = 0$. So numerical tests on minima are easy to perform. If we additionally assume that $U_{eff}|_{\min} = 0$ then the class of possible stability domains narrows considerably. Substitution of (5.37) into (5.34) transforms this constraint to

$$(\lambda^2 + 1)\bar{b}_0(\lambda) = (d_1 + 2)Q_1(\lambda) \quad (5.40)$$

and inequalities (5.38) to

$$\begin{aligned} d_1(\lambda^2 + 1)\bar{b}_0(\lambda) + Q_2(\lambda) &> 0 \\ (\lambda^2 + 1)[Q_2(\lambda) - (\lambda^2 + 1)\bar{b}_0(\lambda)] &\geq \\ \geq (d_1 + 1)(\lambda^2 - 1)^2\bar{b}_0(\lambda) &\geq 0. \end{aligned} \quad (5.41)$$

5.3. Coinciding scale factors ($\lambda = 1$), $R_1, \Lambda \neq 0$

In this case we have to consider roots of both types. Let us start with type I roots. Relations (5.37) read now

$$\begin{aligned} y^{D-2} &= \frac{Q_2(1)}{R_1}, \\ y^D &= \frac{(d_1 + 1)}{\Lambda d_1} [Q_2(1) - 2Q_1(1)] \end{aligned} \quad (5.42)$$

with

$$Q_1(1) = 2 \sum_{i=0}^{d_1} P_i \quad (5.43)$$

$$Q_2(1) = [(\lambda^2 + 1)\bar{b}_0(\lambda)]_{\lambda=1} = 2 \sum_{i=0}^{d_1} P_i(i+2)^2$$

so that at the extremum point the sign relation $\text{sign}(\sum_{i=0}^{d_1} P_i) = \text{sign}(R_1)$ must hold and the corresponding scale factor is simply given as

$$y_0 = \left(\frac{2 \sum_{i=0}^{d_1} P_i}{R_1} \right)^{1/(D-2)}.$$

The minimum conditions (5.38) reduce now to

$$2(d_1 + 2) \sum_{i=0}^{d_1} P_i > \sum_{i=0}^{d_1} P_i(i+2)^2 \quad (5.44)$$

and for $U_{eff}|_{\min} = 0$ even to

$$(d_1 + 2) \sum_{i=0}^{d_1} P_i = \sum_{i=0}^{d_1} P_i(i+2)^2 > 0. \quad (5.45)$$

Substitution of (5.42) into (5.27) shows that for $\lambda = 1$ we have

$$\begin{aligned} \partial_{yy}^2 U_{eff}|_{\lambda=1} &= \partial_{xx}^2 U_{eff}|_{\lambda=1} = \partial_{xy}^2 U_{eff}|_{\lambda=1} \quad (5.46) \\ &= y^{-2D+2}(d_1 + 1)[2(d_1 + 2)Q_1(1) - Q_2(1)]. \end{aligned}$$

This implies $\det(\tilde{A}_c) = 0$ and according to (5.29) the eigenvalues $w_{(c)1,2}$ of the Hessian \tilde{A}_c (5.28) are given as

$$w_{(c)1} = 2\partial_{yy}^2 U_{eff}|_{\lambda=1}, \quad w_{(c)2} = 0. \quad (5.47)$$

The condition $w_{(c)1} > 0$ is equivalent to the inequality (5.44). We note that the degeneracy (5.46), $\det(\tilde{A}_c) = 0$, $w_{(c)2} = 0$ is caused by the special and highly symmetric form of the effective potential for the model with two identical factor-spaces. It is only in a secondary way related to the coinciding scale factors ($x = y$) for $\lambda = 1$. In Appendix C we illustrate this fact with a simple counter-example, i.e., with a potential that gives $w_{(c)1,2} > 0$ for a minimum at $x = y$.

We turn now to the roots of type II given by the polynomial $I_{2+}(\lambda = 1, y)$ (5.21). From the structure of $I_{2+}(\lambda = 1)$ immediately follows:

1. Because $I_{2+}(\lambda = 1)$ contains only terms with even degree in y , there exist no real roots — and hence no extrema of the effective potential U_{eff} — for parameter combinations with:

$$\text{sign} \left(\sum_{i=0}^{d_1} P_i \right) = \text{sign}(\Lambda) \neq \text{sign}(R_1). \quad (5.48)$$

2. For arbitrary parameters $\Lambda, R_1, \bar{\Delta} := \sum_{i=0}^{d_1} P_i$ roots of $I_{2+}(\lambda = 1)$ can be found by analytical methods up to dimensions $d_1 \leq 2$ performing a substitution $z := y^2$ and using standard techniques for polynomials of degree $\deg_z I_{2+}(\lambda = 1) \leq 4$. Because of $R_1 \neq 0 \Leftrightarrow d_1 \geq 2$ such considerations are restricted to the case $d_1 = 2$.

3. There exist no general mathematical methods to obtain roots of polynomials with degree $\deg_z I_{2+}(\lambda = 1) > 4$ and *arbitrary* coefficients analytically. For special restricted classes of coefficients techniques of number theory [21] are applicable. We do not use such techniques in the present paper. For polynomials $I_{2+}(\lambda = 1)$ and dimensions $\dim M_1 = \dim M_2 = d_1 > 2$ this implies that arbitrary parameter sets should be analyzed numerically or parameters $\Lambda, R_1, \bar{\Delta}$ should be fine tuned — chosen ad hoc in such a way that $I_{2+}(\lambda = 1) = 0$ is fulfilled.

In the following we derive a necessary condition for the existence of a minimum of the effective potential with fine-tuned parameters. Using the ansatz

$$\frac{\Lambda d_1}{d_1 + 1} = \sigma_1 y_0^{-2}; \quad \bar{\Delta} := \sum_{i=0}^{d_1} P_i = \sigma_2 y_0^{D-2} \quad (5.49)$$

equation (5.21) reduces to

$$(\sigma_1 - R_1 + 4\sigma_2)y_0^{D-2} = 0. \quad (5.50)$$

Without loss of generality we choose σ_2 as free parameter, and hence $\sigma_1 = R_1 - 4\sigma_2$, so that from relations (5.49)

$$y_0^{D-2} = \frac{\bar{\Delta}}{\sigma_2}, \quad y_0^D = \frac{d_1 + 1}{\Lambda d_1} \bar{\Delta} \left(\frac{R_1}{\sigma_2} - 4 \right) \quad (5.51)$$

and (5.33) minimum condition (5.32) reads

$$\begin{aligned} \partial_{yy}^2 \tilde{U}_{eff} \Big|_{\min} &= \\ &= 4y_0^{-2D+2}(d_1+1) \left[4(d_1+2) - \frac{R_1}{\sigma_2} \right] \bar{\Delta} > 0 \end{aligned} \quad (5.52)$$

or

$$(2D - \frac{R_1}{\sigma_2}) \sum_{i=0}^{d_1} P_i > 0. \quad (5.53)$$

We see that there exists a critical value $\sigma_c = \frac{R_1}{2D}$ which separates stability-domains with different signs of $\bar{\Delta}$. From $y_0^{D-2} > 0$ and (5.49) follows $\text{sign}(\bar{\Delta}) = \text{sign}(\sigma_2)$ so that

$$\begin{aligned} \bar{\Delta} = \sum_{i=0}^{d_1} P_i > 0 &\iff \sigma_2 > \sigma_c, \quad \sigma_2 > 0 \\ \bar{\Delta} = \sum_{i=0}^{d_1} P_i < 0 &\iff 0 > \sigma_2 > \sigma_c. \end{aligned} \quad (5.54)$$

To complete our considerations of the degenerate case ($\lambda = 1$), $R_1, \Lambda \neq 0$ we derive the constraint $U_{eff} \Big|_{\min} = 0$. By use of (5.34) and (5.51) this is easily done to yield $\sigma_2 = R_1/D = 2\sigma_c$. So the constraint fixes the free parameter σ_2 . Remembering that according to our temporary notation $y := a_2 \equiv e^{\beta^2}$ the value y_0 defines the scale factor of the internal spaces at the minimum position of the effective potential, we get now for the fine-tuning conditions (5.49)

$$\begin{aligned} \Lambda &= \frac{(D-2)R_1}{Da_{(c)2}^2}, \quad \bar{\Delta} = \frac{R_1 a_{(c)2}^{D-2}}{D}, \\ \bar{\Delta}^2 &= \frac{R_1^D (D-2)^{D-2}}{D^D \Lambda^{D-2}} \end{aligned} \quad (5.55)$$

— the well-known conditions widely used in literature [27]. From (5.54), (5.55) and $a_{(c)2} > 0$ we see that for $\sigma_2 = R_1/D = 2\sigma_c$ the stability-domain in parameter-space $\mathbb{R}_{par}^{d_1+3}$ is narrowed to the sector

$$\bar{\Delta} = \sum_{i=0}^{d_1} P_i > 0, \quad R_1 > 0, \quad \Lambda > 0. \quad (5.56)$$

5.4. Vanishing curvature-scalars ($R_1 = 0$), $\Lambda \neq 0$

For vanishing curvature scalars equations (5.16) reduce to

$$\begin{aligned} I_{2+} &= -2(D-2)\bar{a}_0(\lambda) - (D-4)\Lambda y^D = 0 \quad (a) \\ I_{2-} &= -2(\lambda^2 - 1)\bar{b}_0(\lambda) = 0. \quad (b) \end{aligned} \quad (5.57)$$

Extrema of the effective potential are given by roots of $I_{2-} = 0$ with scale factors defined as

$$y^D = -\frac{2(d_1+1)\bar{a}_0(\lambda)}{\Lambda d_1} \equiv -\frac{2(d_1+1)Q_1(\lambda)}{\Lambda d_1}. \quad (5.58)$$

Substitution of (5.58) into minimum-conditions (5.30) yields the following inequalities

$$\begin{aligned} Q_1(\lambda) &\geq 0, \quad Q_2(\lambda) \geq 0, \quad Q_1(\lambda) + Q_2(\lambda) > 0 \\ 8(d_1+1)(d_1+2)Q_1(\lambda)Q_2(\lambda) &\geq \\ &\geq (4d_1+5)^2(\lambda^2-1)^2\bar{b}_0^2(\lambda). \end{aligned} \quad (5.59)$$

From (5.58) and (5.59) we see that for even D positive y are only allowed when the bare cosmological constant Λ is negative: $\Lambda < 0$.

As in the case of nonvanishing curvature scalars so also roots of $I_{2-} = 0$ split into two classes. For nondegenerate physical relevant configurations ($\lambda \neq 1$) the corresponding λ_i must satisfy equation

$$\bar{b}_0(\lambda) = \sum_{i=0}^{d_1} P_i (2+i) \lambda^{d_1-i} \sum_{j=0}^{i+1} \lambda^{2j} = 0. \quad (5.60)$$

For $\lambda > 0$ this is only possible when there exist P_i with different signs.

In the case of degenerate configurations ($\lambda = 1$) equation $I_{2-} = 0$ is trivially satisfied and the scale factor at the minimum of the effective potential given by

$$y_0^D = -\frac{4(d_1+1) \sum_{i=0}^{d_1} P_i}{\Lambda d_1} \quad (5.61)$$

with additional condition

$$\begin{aligned} \sum_{i=0}^{d_1} P_i &\geq 0, \quad \sum_{i=0}^{d_1} P_i (2+i)^2 \geq 0, \\ \sum_{i=0}^{d_1} P_i [(2+i)^2 + 1] &> 0. \end{aligned} \quad (5.62)$$

From inequalities (5.59) and (5.62) immediately follows that effective potentials with parameters ($P_0 < 0, \dots, P_{d_1} < 0$) are not stable.

5.5. Vanishing curvature scalars and vanishing cosmological constants ($R_1 = 0$, $\Lambda = 0$)

In this case equations (5.16) contain only the projective coordinate $\lambda = y/x$

$$\begin{aligned} I_{2+} &= -2(D-2)\bar{a}_0(\lambda) = 0 \quad (a) \\ I_{2-} &= -2(\lambda^2 - 1)\bar{b}_0(\lambda) = 0. \quad (b) \end{aligned} \quad (5.63)$$

Corresponding physical configurations are possible for domains in parameter space given by

$$\lambda = 1, \quad \sum_{i=0}^{d_1} P_i = 0 \quad (5.64)$$

or

$$\lambda \neq 1, \quad R_\lambda[\bar{a}_0(\lambda), \bar{b}_0(\lambda)] = 0. \quad (5.65)$$

Minima of the effective potential are localized at lines $\{\lambda_i = y/x\}$ and must be stabilized by additional terms. Otherwise we get an unstable "run-away" minimum of the potential.

5.6. Generalization to n -scale-factor models

The analytical methods used in the above considerations on stability conditions of internal space configurations with two scale factors can be extended to configurations with 3 and more scale factors by techniques of the theory of commutative rings [18]. In this case constraints, similar to polynomial (5.8) $w(\lambda)$, follow from resultant systems on homogeneous polynomials. We note that in master equations (5.5) we can pass from affine coordinates $\{x, y\}$ to projective coordinates $\{X, Y, Z \mid x = X/Z, y = Y/Z\}$ and transform polynomials $I_{1\pm}$ to homogeneous polynomials in $\{X, Y, Z\}$ so that these generalizations are immediately to perform. A deeper insight in extremum conditions can be gained by means of algebraic geometry [19]. Polynomials $I_{1\pm}$ define two algebraic curves on the $\{x, y\}$ -plane and solutions of system (5.5) $I_{1\pm}(x, y) = 0$ correspond to intersection-points of these curves. For n scale factors extremum conditions $\{\partial_{a_i} U_{eff} = 0\}_{i=1}^n$ would result in n polynomials $I_i(x_1, \dots, x_n) = 0$ defining n algebraic varieties on \mathbb{R}^n . The sets of solutions of system $\{I_i(x_1, \dots, x_n) = 0\}_{i=1}^n$, or equivalently, the intersection points of the corresponding algebraic varieties, define the extremum points of U_{eff} .

6. Conclusion

This paper is devoted to the problem of stable compactification of internal spaces in multidimensional cosmological models. This is one of the most important problems in multidimensional cosmology because via stable compactification of the internal dimensions near Planck length we can explain unobservability of extra dimensions. With the help of dimensional reduction we obtain an effective D_0 -dimensional (usually $D_0 = 4$) theory in the Brans-Dicke and Einstein frames. The Einstein frame is considered here as the physical one. Stable compactification is achieved here due to the Casimir effect which is induced by the non-trivial topology of the space-time. The calculation of the Casimir effect in the case of more than one scale factors is a very complicated problem. That is the reason for proposing a Casimir-like ansatz for the energy density of the massless scalar field fluctuations. In this ansatz all internal factors are included on equal footing. The corresponding equation has correct Casimir dimension: $[\text{cm}]^{-D}$ and gives correct transition to the one-scale-factor limit. Stable configurations with respect to the internal scale factor excitations are found in the cases of one and two internal spaces.

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Appendix A: Casimir effect

It is well known that boundaries or nontrivial topology of the space-time induce a vacuum polarization of quantized fields due to changes in the spectrum of the vacuum fluctuations relative to a corresponding unbounded open flat-space model. This phenomenon is known as vacuum Casimir effect and in our case arises due to the compactness of the internal spaces M_i ($i = 0, \dots, n$). In the general case of a quantum field living at finite temperature in a multidimensional cosmological model, the Casimir free energy due to the presence of compactified internal factor-spaces is defined as

$$F_c = F[\{a_i\}_{i=1}^n] - F[\{a_i \rightarrow \infty\}_{i=1}^n], \quad (\text{A.1})$$

where $F[\{a_i\}_{i=1}^n]$ and $F[\{a_i \rightarrow \infty\}_{i=1}^n]$ are the free energy for a compactified (finite-scale-factor) model and a corresponding decompactified (flat) model, respectively. It is clear that this definition is only meaningful, when supplemented with a regularization method yielding finite results for the Casimir energy.

In what follows we briefly discuss some regularization techniques for the Casimir energy of a massive scalar field in a universe with metric (2.1) and give some arguments in favour of an improved version of the Casimir-like potential, proposed in [23], [24].

Let us consider a massive (with the mass M) scalar field Φ with arbitrary coupling to gravity in a universe with the metric (2.1). The Klein-Gordon-Fock equation for the conformally transformed scalar field

$$\Phi = \tilde{\Phi} a_0^{(1-d_0)/2} \quad (\text{A.2})$$

reads [28]

$$\begin{aligned} & \ddot{\tilde{\Phi}} - \Delta[g^{(0)}] \tilde{\Phi} - a_0^2 \sum_{i=1}^n \frac{1}{a_i^2} \Delta[g^{(i)}] \tilde{\Phi} + \\ & + [M^2 a_0^2 + \xi d_0(d_0 - 1)k_0 + \\ & + \xi a_0^2 \sum_{i=1}^n \frac{d_i(d_i - 1)k_i}{a_i^2}] \tilde{\Phi} = 0, \end{aligned} \quad (\text{A.3})$$

where the overdot denotes the derivative with respect to the conformal time η (we choose the time gauge: $e^\gamma = e^{\beta_0} = a_0$), ξ is the coupling constant, M_i ($i = 0, \dots, n$) are the spaces of constant curvature: $R[g^{(i)}] = k_i d_i(d_i - 1)$, $k_i = \pm 1, 0$, and $\Delta[g^{(i)}]$ are the Laplace-Beltrami operators on M_i :

$$\Delta[g^{(i)}] = \frac{1}{\sqrt{|g^{(i)}|}} \frac{\partial}{\partial x^m} \left(\sqrt{|g^{(i)}|} g^{(i)mn} \frac{\partial}{\partial x^n} \right). \quad (\text{A.4})$$

To get the pure Casimir effect without dynamic effects admixture we supposed in the equation (A.3) all scale factors to be static, what implies that the space-time itself becomes ultrastatic.

We denote by x^i the collective spatial coordinates of the i -th space and make for the scalar field the product ansatz

$$\tilde{\Phi} = g(\eta)Y(x^0)\dots Y(x^n). \quad (\text{A.5})$$

Further, we assume that the scalar field on the cosmological background is situated in the eigenstates of the Laplace-Beltrami operators

$$\Delta \left[g^{(i)} \right] Y_i = -\bar{n}_i^2 Y_i, \quad i = 0, \dots, n \quad (\text{A.6})$$

so that (A.3) can be rewritten as

$$\ddot{g} + a_0^2 \left[M^2 + \sum_{i=0}^n \frac{\bar{n}_i^2 + \xi d_i(d_i - 1)k_i}{a_i^2} \right] g = 0.$$

Thus the physical frequency squared of the scalar field reads

$$\omega_{\bar{n}_0, \dots, \bar{n}_n}^2 = M^2 + \sum_{i=0}^n \frac{\bar{n}_i^2 + \xi d_i(d_i - 1)k_i}{a_i^2}. \quad (\text{A.7})$$

The free energy of the scalar field at temperature $k_B T \equiv 1/\beta$ is now given as [22], [29]

$$\begin{aligned} F &= \frac{1}{\beta} \sum_J \ln [1 - \exp(-\beta \hbar \omega_{\bar{n}_0, \dots, \bar{n}_n})] + \\ &+ \frac{\hbar}{2} \sum_J \omega_{\bar{n}_0, \dots, \bar{n}_n}, \end{aligned} \quad (\text{A.8})$$

where J is a collective index for all quantum numbers. The eigenvalues $-\bar{n}_i^2$ depending on the topology of the spaces M_i and the boundary conditions imposed on Y_i can be expressed in terms of the energy level quantum numbers n_i with degeneracy $p_i(n_i)$. For example, $\bar{n}^2 = n^2$ ($n = 0, \pm 1, \pm 2, \dots$), $p = 1$ for S^1 ; $\bar{n}^2 = n(n+1)$ ($n = 0, 1, 2, \dots$), $p = 2n+1$ for S^2 ; $\bar{n}^2 = n^2 - 1$ ($n = 1, 2, 3, \dots$), $p = n^2$ for S^3 and $\bar{n}^2 = n(n+3)$, $p = (n+1)(n+2)(2n+3)/6$ for S^4 . Our further consideration we restrict, for simplicity, to the case of the zero-temperature (vacuum) Casimir effect. Setting $T = 0$ in equation (A.8) we get for the vacuum free energy

$$\begin{aligned} F &= \frac{\hbar}{2} \sum_{n_0} \dots \sum_{n_n} p_0(n_0) \dots p_n(n_n) \omega_{n_0, \dots, n_n} \equiv \\ &\equiv \frac{\hbar}{2} \sum_J \omega_n, \end{aligned} \quad (\text{A.9})$$

so that the Casimir free energy according to (A.1) can be rewritten as

$$F_c = \frac{\hbar}{2} \sum_J \omega_n - \frac{\hbar}{2} \int dJ \omega_n. \quad (\text{A.10})$$

The internal energy density of the fluctuations is found from the relation

$$\rho = \frac{1}{V} \frac{\partial(\beta F_c)}{\partial \beta} = \frac{1}{V} F_c, \quad (\text{A.11})$$

where $V = \prod_{i=0}^n V_i = \prod_{i=0}^n a_i^{d_i} \mu_i$ is the space volume of the universe and

$$\mu_i = \int d^{d_i} y \sqrt{|g^{(i)}|}, \quad i = 0, \dots, n. \quad (\text{A.12})$$

Pressure of fluctuations and Casimir energy-momentum tensor read, respectively,

$$P_i = -\frac{1}{\prod_{j \neq i} V_j} \frac{\partial F_c}{\partial V_i} = -\frac{1}{V d_i} a_i \frac{\partial F_c}{\partial a_i}, \quad i = 0, \dots, n. \quad (\text{A.13})$$

and

$$T_N^M = \text{diag} \left(-\rho, \underbrace{P_0, \dots, P_0}_{d_0}, \dots, \underbrace{P_n, \dots, P_n}_{d_n} \right). \quad (\text{A.14})$$

Let us now turn to the consideration of regularization techniques that are necessary for a meaningful definition of the Casimir energy (A.1), (A.10). Most appropriate for the regularization of multiple sums like (A.9) proved zeta-function technique at the one hand (for an extended review see [17] and cites therein), and the multiple use of the Abel-Plana summation formula on the other hand [16]. For simplicity, we perform our explicit consideration for a manifold M consisting of Ricci-flat (toroidal) factor-spaces $M = \mathbb{R} \times S^1 \times S^1 \times \dots \times S^1$. In this case the physical scalar field frequency reads

$$\omega_J^2 = M^2 + 4\pi^2 \left(\underbrace{\frac{n_{01}^2}{a_0^2} + \dots + \frac{n_{0d_0}^2}{a_0^2}}_{\text{external space}} + \underbrace{\frac{n_1^2}{a_1^2} + \dots + \frac{n_n^2}{a_n^2}}_{\text{internal spaces}} \right) \quad (\text{A.15})$$

and cumulative index and degeneracy are given as $J = \{n_{01}, \dots, n_{0d_0}, n_1, \dots, n_n\}$, $n_{01}, \dots, n_n \in \mathbb{Z}$ and $p_J(J) := p_{01}(n_{01}) \dots p_n(n_n) = 1$. Because of $p_J(J) = 1$ the free energy (A.9) can be expressed in terms of analytically continued inhomogeneous Epstein zeta-functions

$$\begin{aligned} Z_N^M(s; b_1, \dots, b_N) &:= \\ &:= \sum_{n_1, \dots, n_N \in \mathbb{Z}} (M^2 + b_1 n_1^2 + \dots + b_N n_N^2)^{-s} \end{aligned} \quad (\text{A.16})$$

and

$$\begin{aligned} E_N^M(s; b_1, \dots, b_N) &:= \\ &:= \sum_{n_1, \dots, n_N=1}^{\infty} (M^2 + b_1 n_1^2 + \dots + b_N n_N^2)^{-s} \end{aligned} \quad (\text{A.17})$$

directly [17]

$$F = \frac{\hbar}{2} \sum_{n_{01}, \dots, n_n = -\infty}^{\infty} \omega_{n_{01}, \dots, n_n} =$$

$$\begin{aligned}
&= \frac{\hbar}{2} Z_N^M \left(s \rightarrow -\frac{1}{2}; a_0^{-2}, \dots, a_0^{-2}, a_1^{-2}, \dots, a_n^{-2} \right) \\
&= \frac{\hbar}{2} \sum_{j=0}^{N-1} 2^{N-j} \sum_{\{k_1, \dots, k_j\}} E_{N-j}^M \left(s \rightarrow -\frac{1}{2}; \right. \\
&\quad \left. a_0^{-2}, \dots, \widehat{a_{k_1}^{-2}}, \dots, \widehat{a_{k_j}^{-2}}, \dots, a_n^{-2} \right), \quad (\text{A.18})
\end{aligned}$$

where $\{k_1, \dots, k_j\}$ denotes a combination of j indices — whose corresponding terms are absent — out of $N \equiv d_0 + n$ possible ones and we have included the prefactor $4\pi^2$ into the scale factors $a_i/2\pi \rightarrow a_i$ for convenience. (The absent terms $\widehat{a_{k_j}^{-2}}$ result from the zeros of the summation indices $n_{k_j} = 0$ when we change the summation sets from $n_{01}, \dots, n_n \in \mathbb{Z}$ to $n_{01}, \dots, n_n \in \{1, 2, 3, \dots\}$ passing from Z_N^M to E_N^M .) Multiple iterative use of the exact formula [33]

$$\begin{aligned}
E_N^M(s; b_1, \dots, b_N) &= -\frac{1}{2} E_{N-1}^M(s; b_2, \dots, b_N) + \\
&\frac{1}{2} \sqrt{\frac{\pi}{b_1}} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} E_{N-1}^M(s - \frac{1}{2}; b_2, \dots, b_N) + \\
&+ \frac{\pi^s}{\Gamma(s)} b_1^{-\frac{s}{2}} \sum_{k=0}^{\infty} \frac{b_1^{k/2}}{k! (16\pi)^k} \prod_{j=1}^k \left[(2s-1)^2 - (2j-1)^2 \right] \times \\
&\times \sum_{n_1, \dots, n_N=1}^{\infty} n_1^{s-k-1} \times \\
&\times (M^2 + b_2 n_2^2 + \dots + b_N n_N^2)^{-(s+k)/2} \times \\
&\times \exp \left[-\frac{2\pi}{\sqrt{b_1}} (M^2 + b_2 n_2^2 + \dots + b_N n_N^2)^{1/2} \right] \quad (\text{A.19})
\end{aligned}$$

with start condition for the iteration

$$\begin{aligned}
E_1^M(s; 1) &= -\frac{M^{-2s}}{2} + \frac{\sqrt{\pi}}{2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} M^{-2s+1} + \\
&+ \frac{2\pi^s M^{-s+1/2}}{\Gamma(s)} \sum_{p=1}^{\infty} p^{s-1/2} K_{s-1/2}(2\pi p M) \quad (\text{A.20})
\end{aligned}$$

would allow to express the Casimir energy in terms of exponential functions and modified Bessel functions K_ν of the second kind. With these formulae at hand an analysis of the asymptotic behavior of the Casimir energy can be performed. For the case of two scale factors with $a_2 \leq a_1$ the corresponding Epstein zeta functions after an additional regularization have been estimated in Ref. [34] to yield

$$\begin{aligned}
&\sum_{n_1, n_2=1}^{\infty} \sqrt{\left(\frac{n_1}{a_1}\right)^2 + \left(\frac{n_2}{a_2}\right)^2} = \quad (\text{A.21}) \\
&= \frac{1}{24} \left(\frac{1}{a_1} + \frac{1}{a_2} \right) - \frac{\zeta(3)}{8\pi^2} \left(\frac{a_2}{a_1^2} + \frac{a_1}{a_2^2} \right) - \\
&- \frac{\pi^{3/2}}{2\sqrt{a_1 a_2}} \exp \left(-2\pi \frac{a_1}{a_2} \right) [1 + O(10^{-3})]
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n_1, n_2=1}^{\infty} \sqrt{\left(\frac{n_1}{a_1}\right)^2 + \left(\frac{n_2}{a_2}\right)^2} + M^2 = \quad (\text{A.22}) \\
&= \frac{M}{4} - \frac{\pi}{6} a_1 a_2 M^3 + \left(\frac{1}{4\pi} \sqrt{\frac{M}{a_2}} - \frac{M a_1}{4\pi a_2} \right) \times \\
&\times \exp(-2\pi M a_2) [1 + O(10^{-3})].
\end{aligned}$$

To our knowledge, for three and more scale factors similar estimates are performed up to now with less precision (see e.g. [16]). The calculations become more complicated, when one tries to develop analytical approaches for compactified non-Ricci-flat internal factor-spaces with more than one scale factor. In this case the nontrivial degeneracy factor $p_J(J) \neq 1$ in the sums of type (A.9) prevents a direct use of Epstein zeta function methods. Examples for an analytical circumvention of this problem by use of resummation and Mellin transformation are given in [17] for the case of different one-scale-factor-spaces. Generalizations to multi-scale-factor models will need additional efforts in future.

For completeness we make now some brief remarks on the regularization method based on the Abel-Plana summation formula [30], [31], [32]

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n) &= \frac{f(0)}{2} + \int_0^{\infty} f(n) dn + \\
&+ i \int_0^{\infty} \frac{f(i\nu) - f(-i\nu)}{\exp(2\pi\nu) - 1} d\nu \quad (\text{A.23})
\end{aligned}$$

for a function $f(n)$ satisfying some conditions in the complex plane \mathbb{C} . This method was developed mainly in Refs. [15], [16] and consists according to (A.10) in a subtraction of divergent terms containing $\int_0^{\infty} f(n) dn$.

It yields the Casimir energy as finite expression. We applied this method, e.g., to get low-temperature corrections to the vacuum Casimir effect caused by scalar fields living, for example, in a closed Friedmann universe [22] and on a manifold with three-dimensional torus topology $M = \mathbb{R} \times S^1 \times S^1 \times S^1$ [29] (for a consideration of the high temperature regime with the help of zeta function techniques see [35]). Direct application of (A.23) to the triple sum (A.10) gives, e.g., for a massless scalar field [29]

$$\begin{aligned}
F_c &= \frac{4\pi\hbar}{3} \left\{ - \int_0^{\infty} d\nu I\left(\frac{\nu}{a}; 0, 0\right) + \right. \\
&+ \int_0^{\infty} dl \left[\int_{\frac{\hbar}{a}l}^{\infty} d\nu I\left(\frac{\nu}{b}; \frac{l}{a}, 0\right) + \int_{\frac{a}{c}l}^{\infty} d\nu I\left(\frac{\nu}{c}; \frac{l}{a}, 0\right) \right] +
\end{aligned}$$

$$\begin{aligned}
& +2 \sum_{l=0}^{\infty} \left[\int_{\frac{l}{a}}^{\infty} d\nu I\left(\frac{\nu}{b}; \frac{l}{a}, 0\right) + \int_{\frac{l}{a}}^{\infty} d\nu I\left(\frac{\nu}{c}; \frac{l}{a}, 0\right) \right] - \\
& -4 \int_0^{\infty} \int_0^{\infty} dn dm \int_{a\sqrt{\frac{m^2}{b^2} + \frac{n^2}{c^2}}}^{\infty} d\nu I\left(\frac{\nu}{a}; \frac{m}{b}, \frac{n}{c}\right) - \\
& -2 \int_0^{\infty} dn \left(\sum_{l=0}^{\infty} \int_{b\sqrt{\frac{l^2}{a^2} + \frac{n^2}{c^2}}}^{\infty} d\nu I\left(\frac{\nu}{b}; \frac{l}{a}, \frac{n}{c}\right) + \right. \\
& \left. + \sum_{m=0}^{\infty} \int_{a\sqrt{\frac{m^2}{b^2} + \frac{n^2}{c^2}}}^{\infty} d\nu I\left(\frac{\nu}{a}; \frac{m}{b}, \frac{n}{c}\right) \right) - \\
& -4 \sum_{l,m=0}^{\infty} \int_{c\sqrt{\frac{l^2}{a^2} + \frac{m^2}{b^2}}}^{\infty} d\nu I\left(\frac{\nu}{c}; \frac{l}{a}, \frac{m}{b}\right) + C.P. \Bigg\}, \quad (\text{A.24})
\end{aligned}$$

where

$$I\left(\frac{\nu}{a}; \frac{m}{b}, \frac{n}{c}\right) = \sqrt{\frac{\nu^2}{a^2} - \frac{m^2}{b^2} - \frac{n^2}{c^2}} (e^{2\pi\nu} - 1)^{-1}, \quad (\text{A.25})$$

a, b, c are the scale factors corresponding to the wrapped factor-spaces of the three-torus, and $C.P.$ means terms obtained by cyclic permutations $\{a, l; b, m; c, n\}$, for example, $b\frac{n}{c} \rightarrow c\frac{l}{a} \rightarrow a\frac{m}{b}$. Similar to the derivation of (A.21) the symmetric entering of the factor-space scale factors a, b, c into (A.24) will be recovered explicitly only at a final stage of an asymptotic expansion. For some considerations on a possible use of an Abel-Plana summation formula generalized to \mathbb{C}^n we refer to Appendix B.

From the fact that the scale factors for factor-spaces of the same topological type (the same dimension and curvature) up to higher order corrections enter into the physical frequency according to (A.7) in the same way we conclude that there must exist a corresponding interchange symmetry in the expression for the Casimir energy. Asymptotical expansion (A.21) for the toroidal two-scale-factor model shows this symmetry explicitly. Furthermore, from the one-scale-factor model that effectively emerges in the case of coinciding scale factors $a_1 = a_2 = \dots = a_n = a$ we know that the Casimir energy density in this case has the form

$$\rho = C e^{-D\beta} = \frac{C}{a^D}, \quad (\text{A.26})$$

where $D = 1 + d_0 + \sum_{i=1}^n d_i$ is the dimension of the total space-time and C is a dimensionless constant which depends strongly on the topology of the model. It is clear that the Casimir energy density has the same structure for models with only one scale factor [10],

[11], [15], [16], [22] (or all scale factors identically equal each other [12] : $a_0 \equiv a_1 \equiv \dots \equiv a_n \equiv a$). Explicit calculations, formula (A.21) from [34] and asymptotic expansion

$$\rho = -\frac{\pi^2}{90a^4} - \frac{\zeta(3)}{2\pi ab^3} - \frac{\pi}{6abc^2} \quad (\text{A.27})$$

derived in papers [15], [16] for the case $c \geq b \geq a$ show that the Casimir energy density as function of the scale factors has an effective degree $\deg(\rho) = -D$ and in a serial expansion of ρ terms containing $a_i^{-\tilde{D}}$, $\tilde{D} > D$ are compensated by factors containing other scale factors. A sufficiently simple function with this feature and which under decompactification of some of the factor-spaces yields the correct new degree can be constructed as

$$\begin{aligned}
\rho &= e^{-\sum_{i=0}^n d_i \beta^i} \sum_{k_0, \dots, k_n=0}^n \epsilon_{k_0|k_1 \dots k_n} \sum_{\xi_0=0}^{d_{k_1}} \sum_{\xi_1=0}^{d_{k_2}} \dots \\
&\dots \sum_{\xi_{n-1}=0}^{d_{k_n}} A_{\xi_0 \dots \xi_{n-1}}^{(k_0)} \frac{(e^{\beta^{k_1}})^{\xi_0} \dots (e^{\beta^{k_n}})^{\xi_{n-1}}}{(e^{\beta^{k_0}})^{\xi_0 + \xi_1 + \dots + \xi_{n-1} + 1}}, \quad (\text{A.28})
\end{aligned}$$

where $A_{\xi_0 \dots \xi_{n-1}}^{(k_0)}$ are dimensionless constants which depend on the topology of the model and $\epsilon_{ik \dots m} = (-1)^P \varepsilon_{ik \dots m}$. Here $\varepsilon_{ik \dots m}$ is the totally antisymmetric symbol ($\varepsilon_{01 \dots m} = +1$), P is the number of the permutations of the $01 \dots n$ resulting in $ik \dots m$. $|k_1 k_2 \dots k_n|$ means summation is taken over $k_1 < k_2 < \dots < k_n$. Formally, the Eq. (A.28) follows from expression (A.21) if we generalize it to the case of $M = \mathbb{R} \times T^{d_0} \times \dots \times T^{d_n}$, where T^{d_i} are d_i -tori, and omit the exponential terms.

Let M_0 denote the external space and its scale factor $a_0 \rightarrow \infty$. Then equation (A.28) yields

$$\begin{aligned}
\rho &= e^{-\sum_{i=1}^n d_i \beta^i} \sum_{k_1, \dots, k_n=1}^n \epsilon_{k_1|k_2 \dots k_n} \sum_{\xi_1=0}^{d_{k_2}} \dots \\
&\dots \sum_{\xi_{n-1}=0}^{d_{k_n}} A_{\xi_1 \dots \xi_{n-1}}^{(k_1)} \frac{(e^{\beta^{k_2}})^{\xi_1} \dots (e^{\beta^{k_n}})^{\xi_{n-1}}}{(e^{\beta^{k_1}})^{D_0 + \xi_1 + \dots + \xi_{n-1}}}, \quad (\text{A.29})
\end{aligned}$$

where $A_{\xi_1 \dots \xi_{n-1}}^{(k_1)} \equiv A_{d_0 \xi_1 \dots \xi_{n-1}}^{(k_1)}$ and $D_0 = d_0 + 1$. For example, in the case of the two internal spaces and $a_0 \gg a_1, a_2$ Eq. (A.28) reads

$$\rho = \frac{1}{a_1^{d_1} a_2^{d_2}} \left[\sum_{\xi=0}^{d_2} A_{\xi}^{(1)} \frac{a_2^{\xi}}{a_1^{D_0 + \xi}} + \sum_{\xi=0}^{d_1} A_{\xi}^{(2)} \frac{a_1^{\xi}}{a_2^{D_0 + \xi}} \right]. \quad (\text{A.30})$$

It is clear, that for identical spaces $M_1 \equiv M_2$ we have: $d_1 = d_2, R[g^{(1)}] = R[g^{(2)}]$ and $A_{\xi}^{(1)} = A_{\xi}^{(2)}$.

Equations (A.28), (A.29) and (A.30) have correct Casimir dimension: cm^{-D} and give correct transition to the case of one scale factor (A.26). This can be easily seen, for example, from Eq. (A.30) when one of the scale factors goes to infinity. In one's turn equation (A.30) may be obtained from equation (A.29) if $n - 2$ internal scale factors go to infinity. Thus, in contrast to the expression (1.3) for the Casimir energy density proposed in the papers [23], [24] in our case the energy density does not equal to zero if at least one of the scale factors is finite.

In the case of a large external space: $a_0 \gg a_1, \dots, a_n$ from the equations (A.11), (A.13) and (A.29) we can easily derive the equation of state in the external space:

$$P_0 = -\rho, \quad (\text{A.31})$$

because here $F_c = V\rho = (\prod_{i=0}^n \mu_i) e^{\gamma_0} \rho(\beta^1, \dots, \beta^n)$ and $\gamma_0 = \sum_{i=0}^n d_i \beta^i$, and

$$\sum_{i=1}^n d_i P_i = D_0 \rho. \quad (\text{A.32})$$

Appendix B: Abel-Plana summation formula in \mathbb{C}^n

Here we give some arguments for a generalization of the Abel-Plana summation formula (A.23) to the case of several variables. For this purpose we first recall that (A.23) is the result of a contour integration in \mathbb{C} . Given a contour $\gamma = \partial A \subset \mathbb{C}$ as boundary of a rectangle $A = \{z \in \mathbb{C} \mid |Im(z)| \leq R; a \leq Re(z) \leq b\}$ one considers the Cauchy contour integral

$$\oint_{\gamma} f(z) \text{ctg}(\pi z) dz. \quad (\text{B.1})$$

If necessary the contour γ should be deformed appropriately to pass around possible branch cuts of $f(z)$. Adding $\oint_{\partial A_+} f(z) dz - \oint_{\partial A_-} f(z) dz$, where $A_{\pm} = \left\{ z \in \mathbb{C} \mid \begin{array}{l} 0 \leq Im(z) \leq R \\ 0 \geq Im(z) \geq -R \end{array} ; a \leq Re(z) \leq b \right\}$ are the halves of the rectangle $A = A_+ \cup A_-$ located at the upper and the lower complex half-plane, we arrive at

$$\begin{aligned} \sum_{s \in L_A} f(s) &= \int_a^b f(z) dz + \int_{\gamma_+} \frac{f(z) dz}{1 - e^{-2\pi i z}} + \\ &+ \int_{\gamma_-} \frac{f(z) dz}{e^{2\pi i z} - 1}. \end{aligned} \quad (\text{B.2})$$

Here the integration contours are defined as $\gamma_{\pm} = \left\{ z \in \gamma \mid \begin{array}{l} 0 \leq Im(z) \\ 0 \geq Im(z) \end{array} \right\}$ and $L_A = (a, b) \cap \mathbb{Z}$. To get Eq. (A.23) the conditions $|f(iR + t)| \leq \varphi(t) e^{\alpha|R|}$, $\alpha < 2\pi$; $\int_a^b \varphi(t) dt \leq M < \infty$ and $\varphi(t) \xrightarrow{t \rightarrow \infty} 0$ should be satisfied. This means that for an application of the Abel-Plana summation formula to the calculation of the

Casimir energy the physical frequency should be multiplied by an appropriate cutoff function. At the end of the calculations this cutoff can be removed.

Relation (B.2) is well suited for a generalization to a multi-contour integral in \mathbb{C}^n . Following [36, 37] we start with a function $f(z)$ polyholomorphic in the domain $S := A_1 \times A_2 \times \dots \times A_n \subset \mathbb{C}^n$ built from the rectangles $A_i \subset \mathbb{C}_i^1$. Then for $f(z)$ holds the integral representation

$$f(z) = \frac{1}{(2\pi i)^n} \oint_{\partial S} \frac{f(t)}{(t_1 - z_1) \dots (t_n - z_n)} dt_1 \wedge \dots \wedge dt_n, \quad (\text{B.3})$$

where ∂S is defined as $\partial S := \partial A_1 \times \partial A_2 \times \dots \times \partial A_n$. Making use of $\sum_{k=-\infty}^{\infty} \frac{1}{a-k} = \pi \text{ctg}(\pi a)$ we can extend this representation to

$$\begin{aligned} \sum_{\{k_1, \dots, k_n\} \in L_S} f(k_1, \dots, k_n) &= \\ \frac{1}{(2\pi i)^n} \oint_{\partial S} f(z_1, \dots, z_n) \prod_{j=1}^n [\pi \text{ctg}(\pi z_j)] dz_1 \wedge \dots \wedge dz_n \end{aligned} \quad (\text{B.4})$$

with $L_S = S \cap \mathbb{Z}^n$ — the n -dimensional lattice segment contained in the domain S . Proceeding as in the case of one complex variable, a relation generalizing (B.2) can be obtained and convergence criteria can be established. The main task in considerations for functions $f(z_1, \dots, z_n)$ with branch cuts, as in the case of the Casimir energy, consists in nontrivial contour deformations and in keeping control over signs and sign changes of the integrals when moving along the oriented contours.

As in the calculations of the Casimir energy in Refs. [15], [16] step by step calculations with the help of (B.4) lead to expressions containing terms with some functions g of the form $g(\frac{a_i}{a_k} z_l)$. These functions lead to an apparent asymmetry between the scale factors a_i at intermediate stages of the calculation, as explicitly shown, e.g., in (A.24) and [17] also. Nevertheless, final asymptotic expansions can be transformed to expressions symmetric in the scale factors.

Appendix C: Potential with minimum of type $w_{(c)1,2} > 0$ for coinciding scale factors

In this Appendix we show with the help of a simple example that a minimum of an effective potential $U_{eff}(x, y)$ at a point with coinciding scale factors ($x = y$) implies not necessarily a degenerate Hessian \tilde{A}_c (5.28) $\det(\tilde{A}_c) = 0, w_{(c)1} > 0, w_{(c)2} = 0$. As potential we choose

$$U_{eff}(x, y) = f(x)g(y) + h(xy) \quad (\text{C.1})$$

with

$$\begin{aligned} f(x) &= a_- x^{-1} + a_+ x \\ g(y) &= b_- y^{-1} + b_+ y \\ h(xy) &= c_- (xy)^{-1} + c_+ xy. \end{aligned} \quad (\text{C.2})$$

This ansatz is not related to any of the physical potentials considered as approximation to the exact Casimir potential. But we will see that for some coefficient sets $\{a_{\pm}, b_{\pm}, c_{\pm}\}$ it provides a minimum with $w_{(c)1,2} > 0$ for coinciding scale factors $x = y > 0$. Proceeding as in subsection 5.1. we start from the extremum condition $\partial_x U_{eff} = 0$, $\partial_y U_{eff} = 0$ and consider the equation system $x\partial_x U_{eff} \pm y\partial_y U_{eff} = 0$. For $x = y$ this yields

$$\begin{aligned} -(a_- b_- + c_-)y^{-3} + (a_+ b_+ + c_+)y^{-1} &= 0 \\ (-a_- b_+ + a_+ b_-)y^{-1} &= 0 \end{aligned} \quad (\text{C.3})$$

so that for $y > 0$ we have

$$y^4 = \frac{a_- b_- + c_-}{a_+ b_+ + c_+}, \quad a_+ b_- = a_- b_+ \quad (\text{C.4})$$

and the second derivatives of the potential read

$$\begin{aligned} \partial_{xx} U_{eff} &= \partial_{yy} U_{eff} = 2(a_+ b_+ + c_+) + 2a_- b_+ y^{-2} \\ \partial_{xy} U_{eff} &= 2(a_+ b_+ + c_+) - 2a_- b_+ y^{-2}. \end{aligned} \quad (\text{C.5})$$

Hence the eigenvalues of the Hessian \tilde{A}_c are given as $w_{(c)1,2} = \partial_{yy} U_{eff} \pm \partial_{xy} U_{eff}$ or $w_{(c)1} = 4(a_+ b_+ + c_+)$ and $w_{(c)2} = 4a_- b_+ y^{-2}$. Choosing appropriate non-vanishing coefficients $a_{\pm}, b_{\pm}, c_{\pm}$ the minimum conditions $w_{(c)1,2} > 0$ are easy to satisfy so that in the general case the potential U_{eff} can have a nondegenerate minimum at a point with coinciding scale factors $x = y$.

References

- [1] J.A.Wheeler, "Geometrodynamics", Academic, New York, 1962.
- [2] M.Gleiser, S.Rajpoot and J.G.Taylor, *Ann.Phys.* **160**, 299 (1985).
- [3] V.D.Ivashchuk, V.N.Melnikov and A.I.Zhuk, *Nuovo Cim.* **B104**, 575 (1989).
- [4] G.F.R.Ellis, *Gen.Rel.Grav.* **2**, 7 (1971).
- [5] D.D.Sokolov and V.F.Shvartsman, *Sov.Phys. JETP* **39**, 196 (1974).
- [6] H.V.Fagundes, *Phys.Rev.Lett.* **70**, 1579 (1993); *Gen.Rel.Grav.* **24**, 199 (1992).
- [7] M.Lachieze-Rey and J.-P.Luminet, *Phys.Rep.* **254**, 135 (1995).
- [8] V.V.Nikulin and I.R.Shafarevich, "Geometry and Groups", Nauka, Moscow, 1983 (in Russian).
- [9] U.Günther and A.Zhuk, *Phys.Rev.* **D56**, 6391 (1997).
- [10] L.H.Ford, *Phys.Rev.* **D11**, 3370 (1975).
- [11] S.G.Mamaev, V.M.Mostepanenko and A.A.Starobinsky, *Sov.Phys. JETP* **43**, 823 (1976).
- [12] B.S.DeWitt, C.F.Hart and C.J.Isham, *Physica* **96A**, 197 (1979).
- [13] P.Candelas and S.Weinberg, *Nucl.Phys.* **B237**, 397 (1984).
- [14] T.Koikawa and M.Yoshimura, *Phys.Lett.* **B150**, 107 (1985).
- [15] V.M.Mostepanenko and N.N.Trunov, *Sov.Phys. Usp.* **31**, 965 (1989).
- [16] V.M.Mostepanenko and N.N.Trunov, "Casimir effect and its applications", Clarendon, Oxford, 1997.
- [17] E.Elizalde, S.D.Odintsov, A.Romeo, A.A.Bytsenko and S.Zerbini, "Zeta regularization techniques with applications", World Scientific, Singapore, 1994.
- [18] B.L. van der Waerden, *Algebra*, v.1,2, Springer, Berlin, 1971, 1967.
- [19] R.J. Walker, *Algebraic Curves*, Princeton, New Jersey, 1950.
- [20] A.Giveau, M.Porrati and E.Rabinovici, *Phys.Rep.* **244**, 77 (1994).
- [21] P. Bundschuh, *Introduction to number theory*, Springer, Berlin, 1992.
- [22] H.Kleinert and A.Zhuk, *Theor.Math.Phys.* **109**, 1483 (1996).
- [23] F.Acceta, M.Gleiser, R.Holman and E.Kolb, *Nucl.Phys.* **B276**, 501 (1986).
- [24] R.Holman, E.Kolb, S.Vadas and Y.Wang, *Phys.Rev.* **D43**, 995 (1991).
- [25] K.Maeda, *Class.Quantum Grav.*, **3**, 233 (1986).
- [26] A.Zhuk, *Grav. and Cosmol.*, **3**, 24 (1997).
- [27] U.Bleyer and A.Zhuk, *Class. Quantum Grav.* **12**, 89 (1995).
- [28] U.Bleyer and A.Zhuk, *Astron.Nachrichten* **316**, 197 (1995).

- [29] H.Kleinert and A.Zhuk, *Theor.Math.Phys.* **108**, 1236 (1996).
- [30] A.Erdélyi et al. (Eds.), "Higher Transcendental Functions", vol.1 (California Institute of Technology) Bateman H. MS Prorect, McGraw-Hill, New York, 1953.
- [31] M.A.Evgrafov, "Analityc functions", Nauka, Moskow, 1968 (in Russian).
- [32] F.W.J.Olver, "Asymptotics and Special Functions", Academic Press, New York, 1974.
- [33] E.Elizalde, *J.Math.Phys.* **31**, 170 (1990), *J.Phys.* **A22**, 931 (1989).
- [34] E.Elizalde, "Zeta-function regularization techniques for series summation and applications", Proceedings, "Quantum field theory under the influence of external conditions", 60 - 79, Leipzig, 1992.
- [35] K.Kirsten, *J.Phys.* **A24**, 3281 (1991).
- [36] B.A.Fuchs, "Introduction to the theory of analytic functions of several complex variables", Nauka, Moscow, 1962 (in Russian).
- [37] A.K.Cikh, "Multidimensional residues and their applications", Nauka, Novosibirsk, 1988 (in Russian).